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Operators between Approximation Spaces

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Abstract. We study some operator ideals between approximation spaces.

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Introduction

The class of all (bounded linear) operators between arbitrary Banach spaces is denoted by $\mathbb{L}$, while $\mathbb{L}(E,F)$ stands for the space of those operators acting from E into F, equipped with the usual *operator norm*

$$\|S\| = \|S : E \to F\| := \sup\{\|Sx\|_F : \|x\| \leq 1\}.$$

E' denotes the set of all functionals on a Banach space E. The closed unit ball of E' is denoted by U° and the identity map of E is denoted by I_E.

We refer to [11] for definitions and well-known facts about operator ideals.

Let $\mathbb{A}$ be an operator ideal. Then *Space*$(\mathbb{A})$ is the class of all Banach spaces E such that $I_E \in \mathbb{A}$.

An operator $T \in \mathbb{L}(E,F)$ is called *absolutely* (q,p)*-summing* $(1 \leq p \leq q < \infty)$ if there exists a constant $c \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|_F^q\right)^{1/q} \leq c \sup\left\{\left(\sum_{i=1}^{n} |\langle x_i, a\rangle|^p\right)^{1/p} : a \in U^\circ\right\}$$

for every finite family of elements $x_1, \ldots, x_n \in E$. The set of these operators is denoted by $\Pi_{q,p}(E,F)$. For $T \in \Pi_{q,p}(E,F)$ we define $\pi_{q,p}(T) := \inf c$, and then $[\Pi_{q,p}, \pi_{q,p}]$ is a normed operator ideal. We put $[\Pi_{p,p}, \pi_{p,p}] = [\Pi_p, \pi_p]$. Further information is also given in [8] and [11].

An E-valued sequence (x_i) is said to be *absolutely* p*-summable* $(1 \leq p < \infty)$ if $(\|x_i\|_E) \in l_p$. The set of these sequences is denoted by $[l_p, E]$. For $(x_i) \in [l_p, E]$ we define

$$\|(x_i)\|_{[l_p,E]} := \left(\sum_{i=1}^{\infty} \|x_i\|_E^p\right)^{1/p}.$$

An E-valued sequence (x_i) is said to be *weakly p-summable* $(1 \leq p < \infty)$ if $(\langle x_i, a\rangle) \in l_p$ for all $a \in E'$. The set of these sequences is denoted by $[w_p, E]$. For $(x_i) \in [w_p, E]$ we define

$$\|(x_i)\|_{[w_p,E]} := \sup \left\{ \left(\sum_{i=1}^{\infty} |\langle x_i, a\rangle|^p \right)^{1/p} : a \in U^\circ \right\}.$$

Let us recall (see [8, p. 218], or [16, p. 94]) that a Banach space E is said to have *cotype* q, with $2 \leq q < \infty$, if there exists a constant $c \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|x_i\|_E^q \right)^{1/q} \leq c \int_0^1 \|\sum_{i=1}^{n} r_i(t) x_i\| \, dt$$

for all finite families of elements $x_1, \dots, x_n \in E$, where r_i denotes the i-th Rademacher function. It is well-known (see [8, p. 224]) that if E is of cotype q then $I_E \in \Pi_{q,1}(E, E)$.

If $1 < p < \infty$, then the dual exponent p' is determined by $1/p + 1/p' = 1$.

In all what follows almost all definitions concerning approximation spaces are adopted from [13].

An *approximation scheme* (E, A_n) is a Banach space E together with a sequence of subsets A_n such that the following conditions are satisfied:

(i) $A_1 \subseteq A_2 \subseteq \ldots \subseteq E$.

(ii) $\lambda A_n \subseteq A_n$ for all scalars λ and $n = 1, 2, \dots$.

(iii) $A_m + A_n \subseteq A_{m+n}$ for $m, n = 1, 2, \dots$. We put $A_0 := \{0\}$.

Let $1 \leq p < \infty$. Let (E, A_n) be an approximation scheme. If $[w_p, A_n]$ and $[l_p, A_n]$ consist of all A_n-valued sequences of $[w_p, E]$ and $[l_p, E]$, respectively, then we get the approximation schemes

$$([w_p, E], [w_p, A_n]) \qquad \text{and} \qquad ([l_p, E], [l_p, A_n]).$$

Let (E, A_n) be an approximation scheme. For $x \in E$ and $n = 1, 2, \dots$, the n-th *approximation number* is defined by

$$\alpha_n(x, E) := \inf \{\|x - a\|_E : a \in A_{n-1}\}.$$

Let $\sigma > 0$ and $1 \leq u \leq \infty$. Then the *approximation space* E_u^σ, or more precisely $(E, A_n)_u^\sigma$, consists of all elements $x \in E$ such that

$$(n^{\sigma - 1/u} \alpha_n(x, E)) \in l_u,$$

where $n = 1, 2, \dots$. We put

$$\|x\|_{E_u^\sigma} := \|(n^{\sigma - 1/u} \alpha_n(x, E))\|_{l_u} \qquad \text{for} \qquad x \in E_u^\sigma.$$

Then E_u^σ is a Banach space.

Theorem 1 (Representation Theorem (cf [13]))**.** *Let (X, A_n) be an approximation scheme. Then $f \in X$ belongs to X_u^ρ if and only if there exist $a_k \in A_{2^k}$ such that*

$$f = \sum_{k=0}^{\infty} a_k \quad \textit{and} \quad (2^{k\rho}\|a_k\|) \in l_u.$$

Moreover,

$$\|f\|_{X_u^\rho}^{rep} := \inf \|(2^{k\rho}\|a_k\|_X) \in l_u,$$

where the infimum is taken over all possible representations, defines an equivalent quasi-norm on X_u^ρ.

An approximation scheme (E, A_n) is called *linear* if there exist a uniformly bounded sequence of linear projections P_n mapping E onto A_n. Then it follows that

$$\|x - P_{n-1}x\|_E \leq c\alpha_n(x, E)$$

for all $x \in E$ and $n = 1, 2, \ldots$, where

$$c := 1 + \sup_n \|P_n\|_{\mathbb{L}(E,E)}.$$

With the help of the projections

$$Q_k := P_{2^{k+1}-1} - P_{2^k-1}$$

we can formulate the

Theorem 2 (Linear Representation Theorem (cf.[13])). *Let (X, A_n) be a linear approximation scheme. Then $f \in X$ belongs to X_u^ρ if and only if*

$$(2^{k\rho}\|Q_k f\|_X) \in l_u$$

In this case we have

$$f = \sum_{k=0}^{\infty} Q_k f.$$

Moreover,

$$\|f\|_{X_u^\rho}^{lin} := (2^{k\rho}\|Q_k f\|_X)\|_{l_u}$$

is an equivalent quasi-norm on X_u^ρ.

1 (q, p)-summing operators

We state the

Lemma 1. *Let $\rho > 0$ and $1 \leq u \leq r < \infty$. Let (E, A_n) be an approximation scheme. Then*

$$([l_r, E], [l_r, A_n])_u^\rho \subseteq [l_r, E_u^\rho].$$

Proof. Let $x \in ([l_r, E], [l_r, A_n])_u^\rho$ with $x := (x_n)$. Then, by the representation theorem of [13], there exist $x^k \in [l_r, A_{2^k}]$ such that $(2^{k\rho}\|x^k\|_{[l_r,E]}) \in l_u$ and $x = \sum_{k=0}^\infty x^k$ (convergence in $[l_r, E]$). If $x^k := (x_n^k)$, then for $k = 0, 1, \ldots$ and $n = 1, 2, \ldots$ we have

$$x_n = \sum_{k=0}^{\infty} x_n^k,$$

$$(2^{k\rho}\|x_n^k\|_E) \in l_u,$$

and

$$x_n^k \in A_{2^k}.$$

Hence, and also from the representation theorem of [13], we get a constant $c > 0$ such that

$$\|x_n\|_{E_u^\rho} \leq c \left(\sum_{k=0}^{\infty} [2^{k\rho}\|x_n^k\|_E]^u \right)^{1/u}$$

for $n = 1, 2, \ldots$. Therefore, since $1 \leq u \leq r < \infty$, we obtain

$$\begin{aligned}\left(\sum_{n=1}^{\infty}[\|x_n\|_{E_u^\rho}]^r\right)^{1/r} &\leq c\left\{\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}[2^{k\rho}\|x_n^k\|_E]^u\right)^{r/u}\right\}^{1/r} \\ &\leq c\left\{\sum_{k=0}^{\infty}\left(2^{k\rho}\left[\sum_{n=1}^{\infty}\|x_n^k\|_E^r\right]^{1/r}\right)^u\right\}^{1/u} < \infty\end{aligned}$$

and then $x \in [l_r, E_u^\rho]$. Consequently

$$([l_r, E], [l_r, A_n])_u^\rho \subseteq [l_r, E_u^\rho],$$

and the continuity of the inclusion follows from the closed graph theorem.

$\boxed{QED}$

Throughout this section we consider (see [12, p. 39]) the metric isomorphisms

$$S_E^p : \mathbb{L}(l_{p'}, E) \to [w_p, E]$$

and

$$S_E^1 : \mathbb{L}(c_0, E) \to [w_1, E],$$

with $1 < p < \infty$. In both cases, the E-valued sequence (x_i) is identified with the operator $R(\alpha_i) := \sum_{i=1}^{\infty} \alpha_i x_i$. Hence, if (E, A_n) is an approximation scheme, then we have the approximation schemes

$$(\mathbb{L}(l_{p'}, E), (S_E^p)^{-1}([w_p, A_n]))$$

and

$$(\mathbb{L}(c_0, E), (S_E^1)^{-1}([w_1, A_n])),$$

for $1 < p < \infty$. The corresponding approximation spaces will be denoted by $\mathbb{L}(l_{p'}, E)_u^\sigma$ and $\mathbb{L}(c_0, E)_u^\sigma$, respectively.

Next we prove the

Lemma 2. *Let $\mu > 0$ and $1 < p < \infty$. Let (E, A_n) be a linear approximation scheme. Then*

$$\mathbb{L}(l_{p'}, E_\infty^\mu) \subseteq \mathbb{L}(l_{p'}, E)_\infty^\mu$$

and

$$\mathbb{L}(c_0, E_\infty^\mu) \subseteq \mathbb{L}(c_0, E)_\infty^\mu.$$

Proof. We consider the first inclusion, since the proof of the second case is analogous.

Let $T \in \mathbb{L}(l_{p'}, E_\infty^\mu)$. Then $P_{n-1}T \in (S_E^p)^{-1}([w_p, A_{n-1}])$, and therefore

$$\alpha_n(T, \mathbb{L}(l_{p'}, E)) \leq \|T - P_{n-1}T\|_{\mathbb{L}(l_{p'}, E)}$$

for $n = 1, 2, \ldots$, where P_n are the corresponding projections from E onto A_n.

If $x \in l_{p'}$ then

$$\|Tx - P_{n-1}Tx\|_E \leq c\alpha_n(Tx, E)$$

for $n = 1, 2, \ldots$, where

$$c := 1 + \sup_n \|P_n\|_{\mathbb{L}(E,E)}.$$

Hence

$$n^\mu\|Tx - P_{n-1}Tx\|_E \leq cn^\mu\alpha_n(Tx, E) \leq c \sup_n n^\mu\alpha_n(Tx, E) = c\|Tx\|_{E_\infty^\mu}$$

and

$$n^{\mu}\|T - P_{n-1}T\|_{\mathbb{L}(l_{p'},E)} \le c\|T\|_{\mathbb{L}(l_{p'},E_\infty^\mu)}$$

for $n = 1, 2, \dots$.

Combining the observations above, we obtain

$$\begin{aligned}\|T\|_{\mathbb{L}(l_{p'},E)_\infty^\mu} &= \sup_n n^{\mu}\alpha_n(T, \mathbb{L}(l_{p'}, E)) \\ &\le \sup_n n^{\mu}\|T - P_{n-1}T\|_{\mathbb{L}(l_{p'},E)} \le c\|T\|_{\mathbb{L}(l_{p'},E_\infty^\mu)}\end{aligned}$$

$\boxed{QED}$

Now we are ready to establish a general result.

Theorem 3. *Let $\sigma > \tau > 0$ and $1 \le u, v \le \infty$. Let (E, A_n) and (F, B_n) be approximation schemes, and suppose that (E, A_n) is linear. Let $T \in \Pi_{q,p}(E, F)$, with $1 \le p \le q < \infty$. If*

$$T(A_n) \subseteq B_n \qquad for \qquad n = 1, 2, \dots,$$

then $T \in \Pi_{q,p}(E_u^\sigma, F_v^\tau)$.

Proof. We assume that $1 < p < \infty$, since the case $p = 1$ can be treated similarly.
In view of $T(A_n) \subseteq B_n$ for $n = 1, 2, \dots$, we have $T \in \mathbb{L}(E_u^\sigma, F_u^\sigma)$. By Proposition 3 of [13] we get $F_u^\sigma \subseteq F_v^\tau$, and then $T \in \mathbb{L}(E_u^\sigma, F_v^\tau)$.

Since $T \in \Pi_{q,p}(E, F)$, from [11, (17.2.3)] if $\hat{T}$ is defined by

$$\hat{T} : (x_i) \to (Tx_i),$$

then $\hat{T} \in \mathbb{L}([w_p, E], [l_q, F])$.

We have

$$\hat{T}S_E^p((S_E^p)^{-1}([w_p, A_n])) = \hat{T}([w_p, A_n]) \subseteq [l_q, B_n] \qquad \text{for} \qquad n = 1, 2, \dots,$$

and this yields the operator

$$\hat{T}S_E^p : \mathbb{L}(l_{p'}, E)_\infty^\sigma \to ([l_q, F], [l_q, B_n])_\infty^\sigma.$$

We choose ρ with $\sigma > \rho > \tau$ and w with $1 \le w \le q$. Then, using Proposition 3 of [13], we obtain

$$([l_q, F], [l_q, B_n])_\infty^\sigma \subseteq ([l_q, F], [l_q, B_n])_w^\rho.$$

From Lemma 1., we get

$$([l_q, F], [l_q, B_n])_w^\rho \subseteq [l_q, F_w^\rho].$$

Consequently, we have the inclusions

$$([l_q, F], [l_q, B_n])_\infty^\sigma \subseteq [l_q, F_w^\rho] \subseteq [l_q, F_v^\tau].$$

$\boxed{QED}$

Now we also consider (see the comments before Lemma 2) the metric isomorphism

$$S_{E_\infty^\sigma}^p : \mathbb{L}(l_{p'}, E_\infty^\sigma) \to [w_p, E_\infty^\sigma].$$

Hence, by Lemma 2. and the observations above, we have the operators

$$[w_p, E_\infty^\sigma] \xrightarrow{(S_{E_\infty^\sigma}^p)^{-1}} \mathbb{L}(l_{p'}, E_\infty^\sigma) \xrightarrow{J_1} \mathbb{L}(l_{p'}, E)_\infty^\sigma$$

and
$$\mathbb{L}(l_{p'}, E)^{\sigma}_{\infty} \xrightarrow{\hat{T}S^{p}_{E}} ([l_q, F], [l_q, B_n])^{\sigma}_{\infty} \xrightarrow{J_2} [l_q, F^{\tau}_{v}],$$
where J_1 and J_2 denote the corresponding inclusions.

Finally, if $U := J_2 \hat{T} S^p_E J_1 (S^p_{E^{\sigma}_{\infty}})^{-1}$ then
$$U \in \mathbb{L}([w_p, E^{\sigma}_{\infty}], [l_q, F^{\tau}_{v}])$$
is of the form
$$U : (x_n) \to (Tx_n)$$
for $(x_n) \in [w_p, E^{\sigma}_{\infty}]$. Therefore, since $E^{\sigma}_{u} \subseteq E^{\sigma}_{\infty}$ we also have
$$U \in \mathbb{L}([w_p, E^{\sigma}_{u}], [l_q, F^{\tau}_{v}]),$$
and from [11, (17.2.3)] we conclude that $T \in \Pi_{q,p}(E^{\sigma}_{u}, F^{\tau}_{v})$.

Remark 1. We mention that in the case of interpolation spaces, a theorem of the above type goes back to J. Peetre [10].

Remark 2. We observe that the observations above can be obtained in the case of (E, A_n) to be quasicomplemented in the sense of [4].

Remark 3. Other more general approximation spaces are found in [1], [2] and [3].

Now, we give some applications.

Let $1 \le p < \infty$. For any measure space (Ω, Σ, μ) with μ positive we define $L_p(\Omega, \Sigma, \mu)$ to be the space of all (equivalence classes of) Σ-measurable functions such that $\int_{\Omega} |f(\omega)|^p \, d\mu(\omega) < \infty$. Such functions are called *p-integrable*. It is a Banach space with the norm $\|f\|_p := (\int_{\Omega} |f(\omega)|^p \, d\mu(\omega))^{1/p}$. In the important example of the real line equipped with the Lebesgue measure, we simply write $L_p(\mathbb{R})$. Is well-known (see for example [16, p. 98]) that the space $L_p(\Omega, \Sigma, \mu)$ is of cotype $\max(2, p)$.

In the following we consider complex-valued 2π-periodic functions on the real line. Then the periodic analogous of $L_p(\mathbb{R})$ is denoted by $L_p(2\pi)$. Its norm is
$$\|f\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p}.$$
The space $L_p(2\pi)$ is also of cotype $\max(2, p)$.

A *trigonometric polynomial* of degree n is a function t which can be represented in the form
$$t(\xi) = \sum_{|k| \le n} \gamma_k \exp(ik\xi) \qquad \text{for all} \qquad \xi \in \mathbb{R},$$
where $\gamma_{-n}, \ldots, \gamma_n \in \mathbb{C}$ and $|\gamma_{-n}| + |\gamma_n| > 0$. If so, then we write $\deg(t) = n$.

Let $1 < p < \infty$. We denote by T_n the subset of $L_p(2\pi)$ which consist of all trigonometric polynomials such that $\deg(t) \le n$. Then we have the linear approximation scheme $(L_p(2\pi), T_n)$. If $\sigma > 0$ and $1 \le u \le \infty$, we put
$$B^{\sigma}_{p,u}(2\pi) := (L_p(2\pi), T_n)^{\sigma}_{u}.$$
It can be seen from approximation theory that $B^{\sigma}_{p,u}(2\pi)$ are the *Besov function spaces* (see [5], [7]).

We are now prepared to give the

Theorem 4. *Let $\sigma > \tau > 0$, $1 < p < \infty$ and $1 \le u, v \le \infty$. Let $q := \max(2, p)$. Then, the embedding operator $I_{B(2\pi)}$ from $B^{\sigma}_{p,u}(2\pi)$ into $B^{\tau}_{p,v}(2\pi)$ satisfies*
$$I_{B(2\pi)} \in \Pi_{q,1}(B^{\sigma}_{p,u}(2\pi), B^{\tau}_{p,v}(2\pi)).$$

Proof. Since the space $L_p(2\pi)$ is of cotype q, then

$$I_{L_p(2\pi)} \in \Pi_{q,1}(L_p(2\pi), L_p(2\pi)).$$

Hence, from Theorem 3 we obtain

$$I_{B(2\pi)} \in \Pi_{q,1}((L_p(2\pi), T_n)^{\sigma}_u, (L_p(2\pi), T_n)^{\tau}_v)$$

with

$$(L_p(2\pi), T_n)^{\sigma}_u = B^{\sigma}_{p,u}(2\pi) \quad \text{and} \quad (L_p(2\pi), T_n)^{\tau}_v = B^{\tau}_{p,v}(2\pi),$$

and this completes the proof.

$\boxed{QED}$

Remark 4. The $(v, 1)$-summing property for embedding operators between some function spaces, was also studied in a different context in [15].

It is well-know, that every function $f \in L_1(2\pi)$ induces a convolution operator

$$C^f_{op} : g(\eta) \to \int_o^{2\pi} f(\xi - \eta)\, g(\eta)\, d\eta$$

on $C(2\pi)$ and $L_p(2\pi)$ with $1 \leq p \leq \infty$.

Theorem 5. *Let $f \in L_p(2\pi)$ with $1 < p < \infty$. Then*

$$C^f_{op} \in \Pi_{p'}(B^{\rho}_{p',u}(2\pi), B^{\sigma}_{p',w})$$

with $1 \leq u, v \leq \infty$ and $\rho > \sigma > 0$.

Proof. We consider the factorization

$$T^f_{op} = IT^f_{op} : \quad L_{p'}(2\pi) \overset{T^f_{op}}{\to} L_\infty(2\pi) \overset{I}{\to} L_{p'}(2\pi)$$

and, by [12,(1.3.9)], we know that

$$I \in \Pi_{p'}(L_\infty(2\pi), L_{p'}(2\pi)).$$

Since $C^f_{op}(T_n) \subset T_n$ for every n, the result follows from 3. $\boxed{QED}$

Let I be the interval [0,1] and let m be an integer, $m \geq -1$. We consider the orthonormal systems $\{f^{(m)}_n : n \geq -m\}$ of spline functions of order m defined on I (for the definition and main properties see [6]) . This system is a basis in $C(I)$ and $L_p(I)$ for $1 \leq p < \infty$

The best approximation error in $L_p(I)$ for $1 \leq p < \infty$ and in $C(I)$ for $p = \infty$ is defined by

$$E^{(m)}_{n,p}(f) := \inf_{\{a_{-m},\dots,a_n\}} \|f - \sum_{j=-m}^{n} a_j f^{(m)}_j\|_p.$$

Let $0 < \alpha < m + 1 + 1/p, 1 \leq \theta < \infty$. Then $B^{\alpha,m}_{p,\theta}(I)$ denotes the Banach space of all functions which belong to $L_p(I)$ for $1 \leq p < \infty$ and to $C(I)$ for $p = \infty$, equipped with the norm

$$\|f\|_{B^{\alpha,m}_{p,\theta}(I)} := \|f\| + \left(\sum_{n=0}^{\infty} [2^{n\alpha} E^{(m)}_{2^n,p}(f)]^{\theta}\right)^{1/\theta}$$

(see [14]). We have

$$C(I)^{\alpha}_{\theta} = B^{\alpha,m}_{\infty,\theta}(I) \qquad L_p(I)^{\alpha}_{\theta} = B^{\alpha,m}_{p,\theta}(I)$$

for $1 \leq p < \infty$. Using that the imbedding $i : C(I) \hookrightarrow L_p(I)$ is p-summing, from Theorem 3, we obtain

Theorem 6. *The imbedding $j : B^{\alpha,m}_{\infty,\theta}(I) \hookrightarrow B^{\beta,m}_{p,\theta}$ with $\alpha > \beta$ is p-summing.*

2 $\sum_p$-property

Let $1 \leq p < \infty$. An operator ideal $\mathbb{A}$ satisfies the Σ_p-condition if and only if for arbitrary Banach spaces E_n, F_n $(n = 1, 2, \ldots)$ the following holds

$$(\Sigma_p): \text{ If } \; T \in \mathbb{L}\big((\Sigma E_n)_p, (\Sigma F_n)_p\big), \; \text{ and } \; Q_n T P_m \in \mathbb{A}(E_m, F_n)$$

$(m, n = 1, 2, \ldots)$, then

$$T \in \mathbb{A}((\Sigma E_n)_p, (\Sigma F_n)_p).$$

Examples (cf.[9]). The following ideals are injective and surjective and satisfy the $\sum_p$-condition.

(i) weakly compact operators.

(ii) Rosenthal operators.

(iii) Banach-Saks operators.

(iv) Decomposing operators.

Now we can formulate the following

Theorem 7. *Let (X, A_n) be a linear approximation scheme such that A_n is finite dimensional for $n = 1, 2, \ldots$. Let $\mathbb{A}$ be an injective and surjective operator ideal which satisfies the Σ_u-condition for $1 < u < \infty$. Then*

$$(X_u^\rho, \|\cdot\|_{X_u^\rho}^{lin}) \in \mathit{Space}(\mathbb{A})$$

Proof. (a) The surjection Q.

Let E_k be the Banach space A_{2^k} with the norm

$$\|x\|_{E_k} := 2^{k\rho}\|x\|_X \qquad (x \in E_k$$

Let $Q : \left(\sum_{k=0}^\infty E_k\right)_u \longrightarrow X_u^\rho$ be the mapping defined by

$$Q(a_k) := \sum_{k=0}^\infty a_k \qquad ((a_k)) \in \left(\sum_{k=0}^\infty E_k\right)_u$$

By the Representation Theorem the series $\sum_{k=0}^\infty a_k$ is convergent in

$$\left(X_u^\rho, \|\cdot\|_{X_u^\rho}^{rep}\right),$$

therefore in

$$\left(X_u^\rho, \|\cdot\|_{X_u^\rho}^{lin}\right).$$

By the same reason, Q is a surjection.

(b) The injection J.

Let $Q_k := P_{2^{k+1}} - P_{2^{k-1}}$ and let $F_k := Q_k(X)$ be equipped with the norm

$$\|x\|_{F_k} := 2^{k\rho}\|x\|_X \qquad (x \in F_k)$$

Let $J : X_u^\rho \longrightarrow \left(\sum_{k=0}^\infty F_k\right)_u$ be the mapping

$$J(f) := (Q_k(f) \qquad (f \in X_u^\rho)$$

Then, by the Linear Representation Theorem, we obtain

$$\|J(f)\|_{\left(\sum_{k=0}^\infty F_k\right)_u} = \|f\|_{\left(X_u^\rho, \|\cdot\|_{X_u^\rho}^{lin}\right)},$$

so that J is a injection.

Finally, we have the composition

$$E_m \xrightarrow{\overline{Q}_m} \left(\sum_{k=0}^{\infty} E_k\right)_u \xrightarrow{Q} X_u^\rho \xrightarrow{J} \left(\sum_{k=0}^{\infty} F_k\right)_u \xrightarrow{\overline{P}_n} F_n$$

where $\overline{Q}_m(x) := (0, 0, \ldots, 0, x, 0, \ldots)$ where the only nonzero entry is the n-th coordinate, and $\overline{P}_n(x_i) := Q_n(x_n)$. Hence

$$\overline{P}_n J Q \overline{Q}_n = Q_n \in \mathbb{F}(E_m, F_n),$$

($\mathbb{F}$ is the set of finite rank operator) consequently

$$\overline{P}_n J Q \overline{Q}_n = Q_m \in \mathbb{A}(E_m, F_n)$$

$(n, m = 1, 2, \ldots)$ and, by the $\sum_u$-condition, we get

$$JQ = J_{X_u^\rho} Q \in \mathbb{A}(X_u^\rho, X_u^\rho).$$

This implies

$$I_{X_u^\rho} \in \mathbb{A}(X_u^\rho, X_u^\rho)$$

since $\mathbb{A}$ is injective and surjective.

$\boxed{QED}$

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